THERE ARE JUST FOUR SECOND-ORDER QUANTIFIERS

BY

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ABSTRACT

Among the second-order quantifiers ranging over relations satisfying a first-order sentence, there are four for which any other one is bi-interpretable with one of them: the trivial, monadic, permutational, and full second order.

Introduction

The problem of elementary theories of permutation groups was discussed in Vazhenin and Rasin [12], McKenzie [5], Pinus [7], and essentially solved in Shelah [11]. It became clear that this is equivalent to the problem of the expressive power of the quantifier Q_P, ranging over permutations. (Of course in rich enough languages it is equivalent to the second-order quantifier, so the interesting case is of languages with no nonlogical symbols.) After examining [11], J. Stavi doubted the naturality of this quantifier, whereas I was convinced that there are no new quantifiers of this kind. At last he suggested, as explication of "this kind", the family of quantifiers Q_ψ, where $\psi = \psi(r)$ is a first-order sentence with the single predicate r, and $(Q_\psi r)\phi$ means: "There is a relation r satisfying ψ such that ϕ"$\cdots$. Here we prove that up to bi-interpretability there are really only four such quantifiers. It seems that this justifies the preoccupation with Q_P. We define interpretability in a way even weaker than in [11]: Q_{ψ_1} is interpretable in Q_{ψ_2} if there is a *first-order* formula $\theta(\bar{x}, y_1 \cdots, r_1, \cdots)$ such that for any *infinite* set A, and relation R over it, $A \vDash \psi_1[R]$, there are elements $a_1, \cdots \in A$ and relations $S_1, \cdots$ over A, $A \vDash \psi_2[S_i]$, such that $A \vDash (\forall \bar{x}) [R(\bar{x}) \equiv \theta(\bar{x}, a_1, \cdots, S_1, \cdots)]$.

Our proofs give somewhat more than what is required. If Q_X is one of those four quantifiers (see Theorem 2 for details) and Q_ψ, Q_X are bi-interpretable, then

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there is a $\theta(\bar{x}, \bar{y}, r_1, \cdots, r_n)$ interpreting Q_X by Q_ψ with bounded n (that is the bound on n is absolute). No attempt has been made to determine a minimal bound, but notice that if Q_ψ, Q_M are bi-interpretable (Q_M—the monadic quantifier) then by Claim 5H, some $\theta(x, y, r)$ interprets Q_M by Q_ψ.

There are several ways in which we can try to generalize our results and most directions were not investigated.

We can quantify over a pair of relations, e.g. two operations defining a field; but this can be reduced to the previous case.

We can permit finite models, but then we can find a quantifier very strong for models with an even number of elements, and trivial for models with an odd number of elements.

We can have quantifiers ranging over pseudo-elementary classes. That is, $(Q_{\psi(r,s)}r) \cdots$, means "there is an r such that for some s, $\psi(r,s)$ holds, and r satisfies $\cdots$". In this case, our proofs give similar classification, but the equivalence classes of Q_M, Q_P are divided into infinitely many equivalence classes. It is not so difficult to give a complete picture. If we want to find which cardinals can be characterized by a sentence with such quantifiers but with no nonlogical symbol, we are stuck by the independence of, e.g., the function $2^{\aleph_\alpha}$.

Another direction is multi-sorted models. Here the classification depends on n-cardinal theorems (see e.g. [1]) but modulo these, it seems possible to give a classification.

Still another direction is to replace first-order logic by the infinitary logic $L_{\omega_1,\omega}$ (or $L_{\lambda,\omega}$). Here it is reasonable to ignore models of cardinality $< \beth_{\omega_1}$. In this case we have a quantifier Q^{λ}_{II} ranging over all two-place relations of cardinality $< \lambda$, where there is $\psi \in L_{\omega_1,\omega}$ which has a model of cardinality μ iff $\mu < \lambda$. We also have the quantifiers ranging over equivalence relations with $< \lambda$ equivalence classes or with equivalence classes of power $\leq \mu < \lambda$ for some μ, where λ satisfies the condition mentioned for Q^{λ}_{II}. It is easy to define when a quantifier Q_ψ is interpretable by a set of quantifiers and hence when a quantifier and set of quantifiers, or two such sets, are bi-interpretable.

CONJECTURE. *Any Q_ψ is bi-interpretable with a finite set consisting of quantifiers mentioned above.*

The following conjecture seems to imply all others. Let A be a fixed infinite set. For each m-place relation R over A define "$(Q_R r) \cdots$" to mean "there is a relation r over A, $(A, R) \cong (A, r)$ such that $\cdots$"

CONJECTURE. *Any quantifier $(Q_R r)$ is bi-interpretable with a finite set of quantifiers $\{(Q_{E_i} r): i < n\}$ where E_i is an equivalence relation over A.*

NOTATION. Let r, s, t denote predicates (= variables over relations); R, S, T (the corresponding) relations; x, y, z individual variables; and a, b, c, d elements. A bar on any one of them means that it is a finite sequence of this sort. Let ϕ, ψ, θ, χ denote formulae, first-order if not stated otherwise. $\phi = \phi(x_1, \cdots, r_1, \cdots)$ means that $x_1, \cdots$ include all the free variables of ϕ, and $r_1, \cdots$ include all the predicates in ϕ. L denotes first-order language (always with equality). Let $\psi = \psi(r)$ always, r have $n(\psi)$ places, and $L_\psi = L(Q_\psi)$ be language L with the added second-order quantifier $(Q_\psi r) \cdots$ which means "there is an r which satisfies ψ such that $\cdots$". Let $R_\psi(A) = \{R: R$ an $n(\psi)$-ary relation over A, $A \vDash \psi[R]\}$ ($\vDash$ denotes satisfaction). Let $(Q_\psi \bar{r})$ mean $(Q_\psi r_1) \cdots (Q_\psi r_n)$, where $\bar{r} = \langle r_1, \cdots, r_n \rangle$. We shall write $\bar{a} \in A$ instead of $\bar{a} = \langle a_1, \cdots, a_n \rangle$, $a_i \in A$. For any $\bar{a}$, $l(\bar{a})$ is its length, and $\bar{a}_i$ or a_i its i'th element, so $\bar{a} = \langle a_1, \cdots, a_{l(\bar{a})} \rangle$.

Let i, j, k, l, m, n range over natural numbers, $i, j, \alpha, \beta, \gamma, \delta$ over ordinals, and λ, μ, κ over cardinals.

A sequence $\bar{a}$ is without repetitions if $i \neq j$ implies $\bar{a}_i \neq \bar{a}_j$, and $\bar{a}, \bar{b}$ are disjoint if $\bar{a}_i \neq \bar{b}_j$ for any i, j. Let $\mathrm{Eq}_\lambda(A)$ $[\mathrm{Eq}^*_\lambda(A)]$ be the set of equivalence relations over A, with each equivalence class having $< \lambda [\lambda]$ elements. Let e denote an equivalence relation.

DEFINITION 1. Q_{ψ_1} is interpretable in Q_{ψ_2} if there is a formula $\phi(\bar{x}, \bar{y}, \bar{r})$, $l(\bar{x}) = n(\psi_1)$ such that for any infinite A and $R_1 \in R_{\psi_1}(A)$ there are $\bar{a} \in A$, $\bar{R} \in R_{\psi_2}(A)$ such that

$$A \vDash (\forall \bar{x})[R_1(\bar{x}) \equiv \phi(\bar{x}, \bar{a}, \bar{R})].$$

DEFINITION 2. Q_{ψ_1} and Q_{ψ_2} are equivalent if each is interpretable in the other.

LEMMA 1. *If Q_{ψ_1} is interpretable in Q_{ψ_2}, then there is a recursive function F from the formulae of any language L_{ψ_1} into those of L_{ψ_2} such that for any infinite model M and sentence $\theta \in L_{\psi_1}$ (not necessarily first-order)*

$$M \vDash \theta \textit{ iff } M \vDash F(\theta).$$

PROOF. We define $F(\theta)$ for formulae θ, by induction on θ. The only nontrivial case is $\theta = (Q_{\psi_1} r)\chi$. Without loss of generality no variable occurs both in θ and in the interpreting formula ϕ (otherwise change names). Replace in $F(\chi)$ and in

ψ_1 every occurrence of $r(\bar{z})$ by $\phi(\bar{z},\bar{y},\bar{r})$, call the results χ^*,ψ_1^* and let $F(\theta) = (\exists\bar{y})(Q_{\psi_1}\bar{r})(\chi^* \wedge \psi_1^*)$.

Our main result is

THEOREM 2. *Each Q_ψ is equivalent to exactly one of the following quantifiers:*

A) *Q_I—the trivial quantifier,* i.e., *Q_{ψ_I}, $\psi_I = r$, $n(\psi_1) = 0$, so L_{ψ_I} is just first-order logic*

B) *Q_M—the monadic second-order quantifier, i.e., Q_{ψ_M}, $\psi_M = (\forall x)[r(x) \equiv r(x)]$, $n(\psi_M) = 1$,*

C) *Q_P—the permutational second-order quantifier, ranging over permutations of the universe of order two, i.e. Q_{ψ_P},*

$$\psi_P = (\forall x)[f(f(x)) = x]$$

(of course we can quantify over functions instead of relations; equivalently we can quantify over $\mathrm{Eq}_3(A)$)

D) *Q_{II}—the (full) second-order quantifier i.e., $Q_{\psi_{II}}$, $\psi_{II} = (\forall xy)$ $[r(x,y) \equiv r(x,y)]$, $n(\psi_{II}) = 2$.*

The proof is broken into a series of lemmas and claims.

LEMMA 3. *Q_I can be interpreted in Q_M, Q_M can be interpreted in Q_P, and Q_P can be interpreted in Q_{II}. However, none of the converses holds. (In fact, in the negative parts, also the conclusion of Lemma 1 fails.)*

PROOF. The positive statements are immediate. As for the negative statements, let L be a language with no predicates or function symbols (except equality, of course), and L_{ord} be the language of models of order.

We know that in $L_{ord}(Q_I)$ there is no formula (with parameters) defining the class of well-ordering but that there is one in $L_{ord}(Q_M)$. Hence Q_M cannot be interpreted by Q_I.

We know that for every sentence $\phi \in L(Q_M)$, either every infinite model satisfies it or no infinite model satisfies it. As in McKenzie [5] (or Pinus [7], Shelah [11]) this is not true for $L(Q_P)$; Q_P cannot be interpreted by Q_M.

By Shelah [11], if a sentence $\phi \in L(Q_P)$ has a model of cardinality $\geqq \aleph_{\Omega\omega}$ $(\Omega = (2^{\aleph_0})^+)$ then ϕ has models of arbitrarily high power. Of course $L(Q_{II})$ does not satisfy this, hence Q_{II} is not interpretable by Q_P.

LEMMA 4. *If Q_ψ is not interpretable by Q_I then Q_M is interpretable by Q_ψ.*

CLAIM 4A. *Q_M is interpretable by Q_ψ if there is a formula $\phi = \phi(x,\bar{y},\bar{r})$,* *and a set A, $\bar{a} \in A$, $\bar{R} \in R_\psi(A)$ such that $\phi(y, \bar{a}, \bar{R})$ divides A into two infinite sets, that is $|\phi(A, \bar{a}, \bar{R})| \geqq \aleph_0$, $|\neg\, \phi(A, \bar{a}, \bar{R})| \geqq \aleph_0$, where $\phi(A, \bar{a}, \bar{R}) = \{b \in A : A \vDash \phi[b, \bar{a}, \bar{R}]\}$.*

PROOF OF CLAIM 4A. Assuming the existence of such ϕ, by the compactness and Lowenheim-Skolem theorems, for every infinite B there are $\bar{a} \in B$, $\bar{R} \in R_\psi(B)$ such that $|B| = |\phi(B, \bar{a}, \bar{R})| = |\neg\, \phi(B, \bar{a}, \bar{R})|$. By applying a permutation of B for every $B_1 \subseteq B$, $|B_1| = |B - B_1| = |B|$, there are $\bar{a} \in A$, $\bar{R} \in R_\psi(B)$ such that $\phi(B, \bar{a}, \bar{R}) = B_1$. Now for every $C \subseteq B$ there are $B_i \subseteq B$ $i = 1, \cdots, 4$ such that $|B_i| = |B - B_i| = |B|$ and $C = (B_1 \cap B_2) \cup (B_3 \cap B_4)$. Let

$$\theta = \theta(x, \bar{y}^*, \bar{r}^*) = [\phi(x, \bar{y}^1, \bar{r}^1) \wedge \phi(x, \bar{y}^2, \bar{r}^2)] \vee [\phi(x, \bar{y}^3, \bar{r}^3] \wedge \phi(x, \bar{y}^4, \bar{r}^4)].$$

Then as the $\bar{a}_i^*$ range over B, and the $\bar{R}_i^*$ range over $R_\psi(B)$, $\theta(B, \bar{a}^*, \bar{R}^*)$ ranges over the subsets of B.

DEFINITION 3.

A) The sequences $\bar{a}^1$, $\bar{a}^2$ are *similar* over B if $\bar{a}^i = \langle \cdots, a_j^i, \cdots \rangle_{j<k}$ and (i) $a_j^1 = a_l^1$ iff $a_j^2 = a_l^2$; (ii) for $b \in B$, $a_j^1 = b$ iff $a_j^2 = b$.

B) The sequences $\bar{a}^1, \bar{a}^2$ are similar over $\bar{b}$ iff they are similar over $\{\cdots, \bar{b}_i, \cdots\}$.

CLAIM 4B. *If Q_M is not interpretable by Q_ψ then for every formula $\phi(\bar{x}, \bar{y}, \bar{r})$ there is a formula $\theta(z, \bar{y}, \bar{r})$ and $n < \omega$ such that for any A, $\bar{b} \in A$, $\bar{R} \in R_\psi(A)$*

(i) *$A \vDash (\exists^{\leqq n} z)\theta(z, \bar{b}, \bar{R})$ that is $|\theta(A, \bar{b}, \bar{R})| \leqq n$*

(ii) *if $\bar{a}^1, \bar{a}^2$ are similar over*

$$\{\cdots, \bar{b}_i, \cdots\} \cup \theta(A, \bar{b}\bar{R}) \text{ then } A \vDash \phi(a^{-1}, \bar{b}, \bar{R}) \equiv \phi(\bar{a}^2, \bar{b}, \bar{R}).$$

REMARK. In the induction step, only the validity of our claim for the previous case is needed.

PROOF OF CLAIM 4B. We shall prove it by induction on $l(\bar{x})$.

For $l(\bar{x}) = 1$ by Claim 4A (and compactness) for some m,

$$\theta_m = [(\exists^{\leqq m} x)\phi(x, \bar{y}, \bar{r}) \to \phi(z, \bar{y}, \bar{r})] \wedge [(\exists^{\leqq m} x) \neg\, \phi(x, \bar{y}, \bar{r}) \to \neg\, \phi(z, \bar{y}, \bar{r})]$$

satisfies our demands.

Suppose we have proved it for $l(\bar{x}) \leqq l$, and we shall prove it for the case $l(\bar{x}) = l + 1$. Choose any A, $\bar{b} \in A$, $\bar{R} \in R_\psi(A)$ and $\bar{x} = \langle x_1, \cdots, x_{l+1} \rangle$, $\bar{x}^1 = \langle x_1, \cdots, x_l \rangle$, $\bar{y}^1 = \langle x_{l+1}, \bar{y}_1, \cdots \rangle$. For $\phi(\bar{x}^1, \bar{y}^1, \bar{r})$ we have proved the claim, and let $\theta(z, \bar{y}^1, \bar{r})$, n be as mentioned there. Now for any $a \in A$ let $Ex(a = \theta(A, a, \bar{b}, \bar{R}) - \{a, \cdots, \bar{b}_i, \cdots\}$. Thus $|Ex(a)| \leqq n$ always.

Let us show that $\cup_{a \in A} Ex(a)$ is finite. If not, define by induction on $i < \omega$, $a_i \in A - \{a_j : j < i\}$, c_i such that $Ex(a_i) \nsubseteq \cup_{j<i} Ex(a_j)$, and $c_i \in Ex(a_i) - \cup_{j<i} Ex(a_j)$. By Ramsey's theorem we can assume (by replacing the sequence of a_i's and c_i's by a subsequence) that the truth value of $c_i \in Ex(a_j)$ depends only on whether $i = j$, $i < j$ or $i > j$. Clearly $c_i \in Ex(a_i)$, and for $j > i$, $j < \omega$, $c_j \notin Ex(a_i)$. Since $|Ex(a_j)| \leqq n$, clearly there is an $i < n+2$ such that $c_i \notin Ex(a_{n+2})$. Hence $c_i \in Ex(a_j)$ iff $i = j$. Similarly $c_i = c_j$ iff $i = j$; and $a_i \neq c_j$. As the a_i's and c_i's are distinct, we can assume that none of them appear in $\bar{b}$. Let f be a permutation of A which interchanges c_{3i+1} with c_{3i+2}, and takes the other elements of A to themselves. Let $\bar{R}^*$ be the image of $\bar{R}$ by f (so f is an isomorphism from $(A, \bar{R})$ onto $(A, \bar{R}^*)$). Clearly $A \vDash (\forall x)\,[\theta(x, a_i, \bar{b}, \bar{R}) \equiv (x, a_i, \bar{b}, \bar{R}^*)]$ iff f takes the set $\theta(A, a_i, \bar{b}, \bar{R})$ onto itself iff i is divisible by three; thus

$$\chi(y, \bar{b}, \bar{R}, \bar{R}^*) = (\forall x)\,[\theta(x, y, \bar{b}, \bar{R}) \equiv \theta(x, y, \bar{b}, \bar{R}^*)]$$

satisfies the conditions mentioned in Claim 4A, a contradiction. Hence $C = \cup_{a \in A} Ex(a)$ is finite. Let $C = \{c_1, \cdots, c_j\}$, $\bar{c} = \langle c_1, \cdots, c_j \rangle$.

Definition 4. Let us call $\chi(\bar{z})$ complete if it is a conjunction such that for every i,j, $z_i = z_j$ or $z_i \neq z_j$ is a conjunct (and all the conjuncts are of this form).

Let $\chi_i(\bar{x}^1, x, \bar{y}, \bar{z})$ $i = 1, \cdots, k$ be a list of all complete formulae in the displayed variables. By definition of Ex for every i, and $a \in A$

$$\text{(i) } A \vDash (\forall \bar{x}^1)\,[\chi_i(\bar{x}^1, a, \bar{b}, \bar{c}) \rightarrow \phi(\bar{x}^1, a, \bar{b}, \bar{R})]$$

or

$$\text{(ii) } A \vDash (\forall \bar{x}^1)\,[\chi_i(\bar{x}^1, a, \bar{b}, \bar{c}) \rightarrow \neg\, \phi(\bar{x}^1, a, \bar{b}, \bar{R})].$$

For each a let $I(a)$ be the set of i's for which (i) holds.

By Claim 4A, except for finitely many a's, all $I(a)$ are equal (to I). Let C^1 be the set of exceptional a's. It is easy to check that:

(*) if $\bar{a}^1, \bar{a}^2$ are similar over $C^2 = \{\cdots, b_i, \cdots\} \cup C \cup C^1$, then $A \vDash \phi[\bar{a}^1, \bar{b}, \bar{R}] \equiv \phi[\bar{a}^2, \bar{b}, \bar{R}]$; C^2 is finite.

Without loss of generality we cannot replace C^2 by a set of smallest cardinality satisfying (*). Let $n_1 = |C^2|$, and let $\theta_1 = \theta_1(z, \bar{y}, \bar{r})$ say that there are $z_2, \cdots, z_{n_1}$, such that if $\bar{x}^1, \bar{x}^2$ are similar over $\{z, z_2, \cdots, z_{n_1}, \cdots, \bar{y}_i, \cdots\}$, then $\phi(\bar{x}^1, \bar{y}, \bar{r}) \equiv \phi(\bar{x}^2, \bar{y}, \bar{r})$.

Subclaim 4C. *$\theta_1(A, \bar{b}, \bar{R})$ is finite.*

Proof of Subclaim 4C. If not, there are distinct C_i^2, $i < \omega$ satisfying (*), $|C_i^2| = n_1$.

Now w.l.o.g. there is a C^*, $|C^*| < n_1$, such that for any $i < j < \omega$, $C_i^2 \cap C_j^2 = C^*$; this follows by Erdös and Rado [2], but we can also prove it directly. Let $C_i^2 = \{c_{i,1}^2, \cdots, c_{i,n_1}^2\}$, and by Ramsey's theorem [9] there is an infinite $I \subseteq \omega$, such that for $1 \leqq l, k \leqq n_1$, $i < j \in I$, the truth value of $c_{i,l}^2 = c_{j,k}^2$ does not depend on the particular i,j. Without loss of generality $I = \omega$. Let

$$C^* = \{c_{0,k}^2 : c_{0,k}^2 = c_{1,k}^2, \quad 1 \leqq k \leqq n_1\}.$$

By definition of I, $C^* \subseteq C_i^2$ for every i. As $C_0^2 \neq C_1^2$, $|C^*| < n_1$. Let $i < j < \omega$. Then clearly $C^* \subseteq C_i^2 \cap C_j^2$; if equality does not hold let $c \in C_i^2 \cap C_j^2 - C^*$. Thus $c = c_{i,k}^2 = c_{j,l}^2$; since $i < j$, this implies $c_{0,k}^2 = c_{2,l}^2$, $c_{1,k}^2 = c_{2,l}^2$, $c_{0,k}^2 = c_{j,l}^2$. Hence $c_{0,k}^2 = c_{1,k}^2 = c_{j,l}^2 = c$, $c_{0,k}^2 \in C^*$, and $c \in C^*$, a contradiction. So it is proved that w.l.o.g. there is such a C^*, but if $\bar{a}^1$, $\bar{a}^2$ are similar over C^* then they are similar over all C_i^2 except finitely many, and this contradicts the definition of n_1. Thus Subclaim 4C is proved.

Continuation of the Proof of Claim 4B. Let $|\theta_1(A, \bar{b}, \bar{R})| = n_2$.

So $\theta_1(z, \bar{y}, \bar{r})$, n_2 satisfy the demands in Claim 4B except that they depend on $A, \bar{b}, \bar{R}$. By the compactness theorem there are $\theta^i(z, \bar{y}, \bar{r})$, n^i $i = 1, \cdots, k (< \omega)$ such that for any $A, \bar{b} \in A$, $\bar{R} \in R_\psi(A)$ there is an i such that θ^i, n^i satisfy the demands of the claim. Let $\theta^* = \theta^*(z, \bar{y}, \bar{r}) = \vee_i [(\exists^{\leq n^i} u)\ \theta^i(u, \bar{y}, \bar{r}) \wedge \theta^i(z, \bar{y}, \bar{r})]$. Clearly this is the right one, so Claim 4B is proved.

Proof of Lemma 4. Assume Q_M is not interpretable by Q_ψ. Use Claim 4B for $\phi(\bar{x}, r) = r(\bar{x})$, and let θ, n be the θ, n whose existence is proved there. Let $\chi_i(\bar{x}, \bar{z})$ $(l(\bar{z}) = n)$ $i = 1, \cdots, k$ be the complete formulae mentioned in the proof of Claim 4B. Let $I_1, \cdots, I_{2^k}$ be the subsets of $\{1, \cdots, k\}$.

Let

$$\phi^*(\bar{x}, \bar{y}, \bar{z}) = \bigwedge_j \ [y_{2j} = y_{2j+1} \rightarrow \bigvee_{i \in I_j} \ \chi_i(\bar{x}, \bar{z})].$$

For an infinite A, for every $\bar{R} \in R_\psi(A)$ let $\{c_1, \cdots, c_n) \supseteq \theta(A, \bar{R})$.

Let $I = \{i : (\exists \bar{x})\, [\chi_i(\bar{x}, \bar{c}) \wedge r(\bar{x})]\}$, j be such that $I = I_j$. Define $\bar{b}$ such that $\bar{b}_{2p} = \bar{b}_{2p+1}$ iff $p = j$. Then

$$A \vDash \phi^*(\bar{x}, \bar{b}, \bar{c}) \equiv r(\bar{x}),$$

a contradiction. Thus Lemma 4 is proved.

Lemma 5. *If Q_ψ is not interpretable by Q_M then Q_P is interpretable by Q_ψ.*

PROOF. Clearly Q_ψ is *a fortiori* not interpretable by Q_I, hence by Lemma 4, Q_M is interpretable by Q_ψ.

CLAIM 5A. *Q_P is interpretable by Q_ψ if there is a formula $\phi(x,y,\bar{z},\bar{r})$, a set A, $\bar{c}\in A$, $\bar{R}\in R_\psi(A)$, $B\subseteq A$ such that $\phi(x,y,\bar{c},\bar{R})$ defines on B an equivalence relation with infinitely many equivalence classes with $\geqq 2$ elements.*

PROOF OF CLAIM 5A. The proof is similar to that of Claim 4A. By replacing B by a subset, we may assume that each equivalence class has exactly two elements and that $A-B$ is infinite. Now for every infinite A, by the compactness and the Lowenheim-Skolem theorems, there are $B\subseteq A$, $\bar{a}\in A$, $\bar{R}\in R_\psi(A)$, such that $|B|=|A-B|=|A|$, and $\phi(x,y,\bar{a},\bar{R})$ defines on B a relation $\in \mathrm{Eq}_2^*(B)$. We can easily find $\bar{b}\in A$, $\bar{S}\in R_\psi(A)$ such that $\phi(x,y,\bar{b},\bar{S})$ defines on $A-B$ an equivalence relation from $\mathrm{Eq}_2^*(A-B)$. Also there is a formula $\phi^*(x,\bar{c},\bar{T})$ $\bar{c}\in A$, $\bar{T}\in R_\psi(A)$, which defines B. So

$$\theta(x,y,\bar{a},\bar{b},\bar{c},\bar{R},\bar{S},\bar{T}) = [\phi^*(x,\bar{c},\bar{T})\equiv\phi^*(y,\bar{c},\bar{T})]$$
$$\wedge\ [\phi^*(x,\bar{c},\bar{T})\rightarrow\phi(x,y,\bar{a},\bar{R})]\wedge[\neg\,\phi^*(x,\bar{c},\bar{T})\rightarrow\phi(x,y,\bar{b},\bar{S})]$$

defines a relation from $\mathrm{Eq}_2^*(A)$.

Clearly for every $e\in\mathrm{Eq}_2^*(A)$ there are $\bar{a}',\bar{b}',\bar{c}'\in A$, $\bar{R}',\bar{S}',\bar{T}'\in R_\psi(A)$, such that

$$A\vDash(\forall xy)\,[\theta(x,y,\bar{a},\cdots)\equiv xey].$$

Since we can interpret Q_M in Q_ψ, by a small change in θ we can have the same for $e\in\mathrm{Eq}_3(A)$. This proves the claim.

DEFINITION 5. We call $\phi=\phi(x_1,\cdots,x_n,r)$ atomic if $\phi=[x_i=x_j]$ or $\phi=r(x_{i_1},\cdots x_{i_{n(\psi)}})$.

DEFINITION 6. For every A, $B\subseteq A$, $R\in R_\psi(A)$, define the equivalence relation $e=e(R,B,A)$ over B by *bec iff* b, $c\in B$, and for every atomic $\phi(x_1,\cdots,x_n)$ and $a_2,\cdots,a_n\in A-B$, $A\vDash\phi[b,a_2,\cdots,R]\equiv\phi[c,a_2,\cdots,R]$.

CLAIM 5B. *$e(R,B,A)$ is defined by a formula in A (with R and B as parameters).*

PROOF. Immediate.

CLAIM 5C. *If Q_F is not interpretable by Q_ψ, then for every $A,B\subseteq A,R\in R_\psi(A)$, $e(R,B,A)$ has finitely many equivalence classes.*

PROOF. Suppose $e(R,B,A)$ has infinitely many equivalence classes. By Claim 5A, only finitely many of them have $\geqq 2$ elements. But if we replace B by a smaller set, $e(R,B,A)$ becomes finer (i.e., the equivalence classes become smaller). Hence w.l.o.g. each equivalence class of $e(R,B,A)$ has one element, and of course B is infinite.

Let f be a permutation of order two of A, such that $f(a) = a \leftrightarrow a \notin B$. Define

$$R_1 = \{\langle a_1, \cdots \rangle : a_1, \cdots \in A,\ \langle f(a_1), \cdots \rangle \in R\}.$$

Let

$$e_1 = \{\langle c, b\rangle : b, c \in B, \text{ for every atomic } \phi(x, \bar{y}, r) \text{ and every } \bar{a} \in (A - B);\ A \vDash \phi[c, \bar{a}, R] \equiv \phi[b, \bar{a}, R_1]\ \ A \vDash \phi[b, \bar{a}, R] \equiv \phi[c, \bar{a}, R_1]\}.$$

It is easy to see that $c = f(b)$, $c, b \in B$ implies $\langle c, b\rangle \in e_1$. It is easy to check that $\langle c, b\rangle \in e_1$ implies $\langle c, f(b)\rangle \in e(R_1, B, A)$ but this implies $c = f(b)$.

Hence $[\langle x, y\rangle \in e_1] \vee x = y$ defines an equivalence relation of $\mathrm{Eq}_2^*(B)$, and clearly it is definable by a formula. By Claim 5A this leads to a contradiction, hence 5C is proved.

Claim 5D. *If Q_P is not interpretable by Q_ψ, then there is a formula $\phi(x, y, r)$ such that for every A, $R \in R_\psi(A)$.*

(i) *$\phi(x, y, R)$ defines an equivalence relation with finitely many equivalence classes.*

(ii) *$A \vDash \phi[a, b, R]$ implies that there is a finite B such that $\langle a, b\rangle \in e(R, B, A)$.*

Proof. Define for A, $R \in R_\psi(A)$ $n < \omega$ the relation

$$e_n(R, A) = \{\langle c, b\rangle : c, b \in A, \text{ there is } B \subseteq A,\ |B| \leqq n \text{ such that } \langle c, b\rangle \in e(R, B, A)\}.$$

Define $\phi_n(x, y, r)$ such that $A \vDash \phi_n[c, b, R]$ iff $\langle c, b\rangle \in e_n(R, A)$, $R \in R_\psi(A)$. Note that $\phi_{n+1}(x, y, r) \rightarrow \phi_n(x, y, r)$ always.

Clearly $e^*(R, A) = \bigcup_{n<\omega} e_n(R, A)$ is an equivalence relation over A. Moreover it has only finitely many equivalence classes. Otherwise choose nonequivalent a_i $1 \leqq i < \omega$. By Claim 5C and the compactness theorem, there is $n_0 < \omega$ such that $e(R^1, B, A)$ always has $\leqq n_0$ equivalence classes, for $B \subseteq A$, $R^1 \in R_\psi(A)$. Let $B = \{a_i : 1 \leqq i \leqq n_0 + 1\}$. Then $e(R, B, A)$ has $n_0 + 1$ equivalence classes (by the choice of the a_i's and the definition of e^*). We prove in fact that $e^*(R, A)$ has $\leqq n_0$ equivalence classes for any $R \in R_\psi(A)$. Hence in

$$\Gamma = \{\psi(r)\} \cup \{\neg\, \phi_n(x_i, x_j, r) : n < \omega,\ 1 \leqq i < j \leqq n_0 + 1\}$$

there is a contradiction.

Thus for some $n_1 < \omega$ there is a contradiction in

$$\{\psi(r)\} \cup \{\neg \phi_n(x_i, x_j, r): n < n_1,\ 1 \leqq i < j \leqq n_0 + 1\}.$$

The closure of $\phi_{n_1}(x, y, r)$ to an equivalence relation is

$$\phi(x, y, r) = {}^{\mathrm{df}}(\exists z_1, \cdots, z_m) \left[\bigwedge_{i=1}^{m} \phi_{n_1}(z_i, z_{i+1}, r) \wedge z_0 = x \wedge z_m = y \right]$$

where $m = 3n_0$ is sufficient. This is because for every $A, R \in R_\psi(A)$ there is a maximal set $\{a_i: 1 \leqq i < i_0\}$ such that $i < j < i_0$ implies $A \vDash \neg \phi_{n_1}(a_i, a_j, R)$; hence $i_0 \leqq n_0$ by the definition of n_1. By the maximality of the set, for every $a \in A$ for at least one i $A \vDash \phi_{n_1}(a, a_i, R)$. Now if b, c are equivalent in the closure of $e_{n_1}(R, A)$ then there are $d_1, \cdots, d_m$, $d_1 = b$, $d_m = c$ and $\langle d_i, d_{i+1} \rangle \in e_{n_1}(R, A)$. Choose such d_i's with minimal m; we should show $m \leqq 3n_0$. For this it suffices to prove there are no four d_i from one $\phi_{n_1}(A, a_i, R)$. Let $1 \leqq i_1 < i_2 < i_3 < i_4 \leqq m$, $d_{i_1}, \cdots, d_{i4} \in \phi_{n_1}(A, a_j, R)$. Then $\langle d_{i_1}, a_j \rangle$, $\langle a_j, d_{i_4} \rangle \in e_{n_1}(R, A)$, hence also $d_1, \cdots, d_{i_1}$, $\bar{a}_j$, $d_{i_4} \cdots, d_m$ is a suitable sequence, and it has smaller length, a contradiction. '

Since $e^*(R, A)$ is an equivalence relation, it refines the closure of $e_{n_1}(R, A)$. Hence $R \in R_\psi(A)$, $A \vDash \phi[b, c, R]$ implies that there is a finite $B \subseteq A$ such that $\langle b, c \rangle \in e(R, B, A)$.

Claim 5E. *In Claim 5D we conclude also that there are $\theta(z, x, y, r)$, $n_2 < \omega$ such that for any A, $R \in R_\psi(A)$, $b, c \in A$,*

(i) $A \vDash (\forall xy)(\exists^{\leqq n_2} z)\ \theta(z, x, y, R)$

(ii) $A \vDash (\forall xyz)[\theta(z, x, y, R) \to z \neq x \wedge z \neq y]$

(iii) $A \vDash \phi[b, c, R]$ *implies* $\langle b, c \rangle \in e(R, B, A)$ *where* $B = \theta(A, b, c, R) \cup \{b, c\}$

(iv) $A \vDash \neg \phi(b, c, R]$ *implies* $A \vDash (\forall z) \neg \theta(z, b, c, R)$.

Proof. By the compactness theorem and Claim 5D, there is an $n_3 < \omega$ such that $R \in R_\psi(A)$, $A \vDash \phi[b, c, R]$ implies $\langle b, c \rangle \in e_{n_3}(R, A)$.

Let $\theta(z, x, y, r)$ say "$\phi(x, y, r)$, $z \neq x$, $z \neq y$ and for some $n \leqq n_3$ there are no $z_1, \cdots, z_{n-1}$ such that $\langle x, y \rangle \in e(r, \{x, y, z_1, \cdots, z_{n-1}\})$, but there are $z_1, \cdots, z_n$ such that $\langle x, y \rangle \in e(r, \{x, y, z_1, \cdots, z_n\}$, and $z = z_1$". As in the proof of Claim 4C for all $R \in R_\psi(A)$, $b, c \in A$, $\theta(A, b, c, R)$ is finite, and so clearly the claim holds.

Claim 5F. *In the conclusion of Claim 5E we can add*

(v) *there is $n_4 < \omega$ such that for $R \in R_\psi(A)$*

$$A \vDash (\exists^{\leqq n_4} z)\ (\exists xy)\theta(z, x, y, R).$$

For this it suffices to prove Claim 5G (by applying Claim 5G twice we get Claim 5F).

CLAIM 5G. *If Q_P is not interpretable by Q_ψ, and for any $R \in R_\psi(A)$, $A \models (\forall \bar{x})(\forall y)(\exists^{\leq m_1} z)\, \theta(z, y, \bar{x}, R)$ and $\theta(z, y, \bar{x}, r) \rightarrow z \neq y$, then for some $m_2 < \omega$, for every $R \in R_\psi(A)$*

$$A \models (\forall \bar{x})(\exists^{\leq m_2} z)(\exists y)\, \theta(z, y, \bar{x}, R).$$

PROOF. If not, by the compactness theorem, there are A, $R \in R_\psi(A)$, $\bar{a} \in A$ such that

(1) $$A \models (\forall y)(\exists^{\leq m_1} z)\ \theta(z, y, \bar{a}, R)$$

(2) for every finite $B \subseteq A$ there are $b \in A$, $c \in A - B$, $A \models \theta(c, b, \bar{a}, R)$.

Define by induction on n, $b_n \in A$, $c_n \in A - \{c_i : i < n\}$ such that $A \models \theta[c_n, b_n, \bar{a}, R]$.

By Ramsey's theorem [9] we can assume that the truth value of $A \models \theta[c_m, b_n, \bar{a}, R]$, $b_n = c_m$ depends only on whether $m = n$, $m < n$ or $m > n$. Since, $A \models (\exists^{\leq m_1} z)\theta(z, b_n, \bar{a}, \bar{R})$ clearly $A \models \theta[c_m, b_n, \bar{a}, R]$ if $m = n$ (reccal that the c_n's are distinct); therefore, b_n's are distinct. Also $b_n \neq c_m$ because (1) if $n = m$, this holds by the assumption on θ, (2) if $n < m$, then $c_1 = b_0 = c_2$, a contradiction, and (3) if $n > m$, $c_1 = b_3 = c_2$, a contradiction.

Also w.l.o.g. $b_n \neq \bar{a}_i$, $c_n \neq \bar{a}_i$, $\models \neg\, \theta[c_n, c_m, \bar{a}, R] \wedge \neg\, \theta(b_n, b_m, \bar{a}, R]$ for $n \neq m$ (otherwise omit finitely many $\langle c_i, b_i \rangle$'s). Let

$$B = \{b_n : n < \omega\} \cup \{c_n : n < \omega\}.$$

Now the formula $y = z \vee \theta(z, y, \bar{a}, R) \vee \theta(y, z, \bar{a}, R]$ defines on B a relation of $\mathrm{Eq}_2^*(B)$, a contradiction. Thus Claim 5G, and hence Claim 5F are proved.

CLAIM 5H. *If Q_P is not interpretable by Q_ψ, then for every A, $R \in R_\psi(A)$, $e^+(R, A) = \{\langle a, b \rangle : a, b \in A$, the permutation f ($f(a) = b$, $f(b) = a$, $f(c) = c$ for $c \neq a, b$) is an automorphism of $(A, R)\}$ is an equivalence relation with finitely many equivalence classes.*

PROOF. Define by induction on n, $1 \leq n < \omega$, formulae

$$\phi_n(x, y, r),\ \theta_n(z, r) \text{ such that}$$

1) for any $R \in R_\psi(A)$, $\phi_n(x, y, R)$ is an equivalence relation with $< k_1(n) < \omega$ equivalence classes

2) for any $R \in R_\psi(A)$, $|\theta_n(A, R)| \leq k_2(n) < \omega$

3) for any $R \in R_\psi(A)$, $a, b \in A$, $A \models \phi_n[a, b, R]$ implies $\langle a, b \rangle \in e(R, (B_n - B_{n-1}) \cup \{a, b\}, A)$

4) for any $1 \leqq n \leqq m < \omega$, $\theta_n(A,R) \subseteq \theta_m(A,R)$ where $B_0 = \emptyset$, $B_n = \theta_n(A,R)$.

For $n = 1$ the existence of ϕ_1, θ_1 follows from Claims 5D, 5E, and 5F and the compactness theorem. (Take $\phi_1 = \phi$, $\theta_1 = (\exists xy)\theta(z,x,y,r)$.)

Suppose $\phi_n \theta_n$ are defined. Let $c_1, \cdots, c_k [k = \sum_{l=1}^{n} k_2(l)]$ be individual constants, and replace $\psi(r)$ by

$$\psi(r) \wedge (\forall z)\left[\bigvee_{l=1}^{n} \theta_l(z,r) \equiv \bigvee_{l=1}^{k} z = c_l\right].$$

Now repeat the proof of Claims 5D, E and F (the change from r to r and c's is technical; just add more atomic formulae). Hence we get ϕ_{n+1} θ_{n+1} as we got ϕ_1 θ_1. Clearly (1), (2) and (3) hold.

Now for any $R \in R_\psi(A)$ define

$$e' = \{\langle a,b\rangle : (\forall n < \omega) A \vDash \phi_n[a,b,R]\}.$$

Clearly e' is an equivalence relation with $\leqq 2^{\aleph_0}$ equivalence classes.

It is also clear that $e^+(R,A)$ is an equivalence relation. We shall now show that if $a\ e'b$, $a,b \notin \cup_n B_n$ and their e'-equivalence class is infinite, then $a\ e^+(R,A)b$.

This implies that $e^+(R,A)$ has $\leqq 2^{\aleph_0}$ equivalence classes, hence by the compactness theorem this is sufficient. For proving that the permutation interchanging a, b is an automorphism, it suffices to prove that if $\phi(x,y,z_1,\cdots,z_m; r)$ is atomic, $c_1,\cdots,c_m \in A - \{a,b\}$, $\vDash \phi(a,b,c_1,\cdots,c_m,r) \equiv \phi(b,a,c_1,\cdots,c_m)$. We can choose n such that $(B_{n+1} - B_n) \cap \{c_1,\cdots,c_m,a,b\} = \emptyset$ and a_1 such that $a_1\ e'a$, $a_1 \notin B_{n+1} \cup \{c_1,\cdots,c_m,a,b\}$. By (3)

$\vDash \phi[a,\ b,c_1,\cdots,\ c_m,r] \equiv \phi[a_1,b,c_1,\cdots,c_m,r]$,

$\vDash \phi[a_1,b,c_1,\cdots,c_m,r] \equiv \phi[a_1,a,c_1,\cdots,c_m,r]$ and also

$\vDash \phi[a_1,a,c_1,\cdots,c_m,r] \equiv \phi[b,a,c_1,\cdots,c_m,r]$. Combining we get the result.

Proof of Lemma 5. From Claim 5H and the compactness theorem, it follows that if Q_p is not interpretable by Q_ψ then there is some $n_5 < \omega$ such that for any A, $R \in R_\psi(A)$, $e^+(R,A)$ has $\leqq n_5$ equivalence classes. Let us show that this implies that Q_ψ is interpretable by Q_M. This implies that for every $A, R \in R_\psi(A)$, there are sets $B_1,\cdots,B_{n_5}$ (the $e^+(R,A)$ equivalence classes) such that the truth value of $R[a_1,\cdots,a_{n(\psi)}]$ $(a_i \in A)$ depends only on the truth values of $a_i = a_j$, $a_i \in B_k$; hence there is a (quantifier free) formula ϕ such that

$$A \vDash (\forall \bar{x})\ [R(\bar{x}) \equiv \phi(\bar{x},B_1,\cdots,B_{n_5})].$$

From the construction, the number of possible ϕ's is finite, and let them be $\phi_1, \cdots, \phi_{n_6}$. Let

$$\phi^* = \bigwedge_{i=1} [y_0 = y_i \rightarrow \phi_i(\bar{x}_1, X_1, \cdots, X_{n_5})]$$

(X_i-variables over sets).

Hence for every infinite A, and $R \in R_\psi(A)$ there are $c_0, \cdots, c_{n_6}$, $B_1, \cdots, B_{n_5}$ such that

$$A \models (\forall \bar{x}) [R(\bar{x}) = \phi^*(\bar{x}, \bar{c}, B_1, \cdots)].$$

Thus the proof of Lemma 5 is complete.

Lemma 6. *If Q_ψ is not interpretable by Q_P then Q_{II} is interpretable by Q_ψ.*

Proof. As Q_ψ is not interpretable by Q_P, it is obviously not interpretable by Q_M; hence by Lemma 5, Q_P is interpretable by Q_ψ.

Definition 7.

1) A family of sequences of length n is pseudofinite if there is a finite set such that in every sequence of the family appears an element from the finite set.

2) A family F of sequences of length n from a model $(A, \bar{R})$ is $\phi(\bar{x}, \bar{y}, \bar{r})$-minimal in $(A, \bar{R})$ $(l(\bar{x}) = n)$ if it is not pseudo-finite, but for any $\bar{a} \in A$, $\{\bar{b} \in F: A \models \phi[\bar{b}, \bar{a}, \bar{R}]\}$ is pseudo-finite *or* $\{\bar{b} \in F: A \models \neg\phi(\bar{b}, \bar{a}, \bar{R})\}$ is pseudo finite.

3) $\phi(x, \bar{a}, \bar{R})$ is algebraic (in $(A, \bar{R})$) if $|\phi(A, \bar{a}, \bar{R})| < \aleph_0$.

4) $\phi(\bar{x}, \bar{a}, \bar{R})$ is pseudo-algebraic (in $(A, \bar{R})$) if $\{\bar{b} \in A: A \models \phi[\bar{b}, \bar{a}, \bar{R}]\}$ is pseudo-finite.

5) $a(\bar{a})$ is (pseudo-) algebraic over B in $(A, \bar{R})$ if for some (pseudo-)algebraic $\phi(x, \bar{b}, \bar{R})$ $(\phi(\bar{x}, \bar{b}, \bar{R}))$, $A \models \phi[a, \bar{b}, \bar{R}]$ $(A \models \phi[\bar{a}, \bar{b}, \bar{R}])$ and $\bar{b} \in B$.

6) The type of $\bar{b}$ over B in $(A, \bar{R})$ is $\{\phi(\bar{x}, \bar{c}, \bar{R}): \bar{c} \in B, A \models \phi[\bar{b}, \bar{c}, \bar{R}]\}$.

Claim 6A. *Q_{II} is interpretable by Q_ψ if there are $\phi(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ $[l(\bar{x}) = l(\bar{y}) = n]$, A, $\bar{R} \in R_\psi(A)$, $\bar{c} \in A$, $B \subseteq A$ such that $\phi(\bar{x}, \bar{y}, \bar{c}, \bar{R})$ defines over ${}^nB = \{\bar{b}: \bar{b} \in B, l(\bar{b}) = n\}$ an equivalence relation, with infinitely many non-pseudo-finite equivalence classes.*

Proof. For $n = 1$, we can show as in Claim 4A, Claim 5A that we can interpret the quantifier over equivalence relations. By Rabin [8], it then follows that we can interpret Q_{II}.

Now we shall reduce the case $n > 1$ to $n = 1$, using the interpretability of Q_P by Q_ψ.

Choose by induction on $\max\{i,j\}$ sequences $\bar{a}^{i\,j}i,j<\omega$ such that

1) $\bar{a}^{i,j}\in B$

2) $A\models\phi[\bar{a}^{i,j},\bar{a}^{l,k};\bar{c},\bar{R}]$ iff $i=l$

3) for $\langle i,j\rangle\neq\langle l,k\rangle$, $\bar{a}^{i,j}$, $\bar{a}^{l,k}$ are disjoint, and $\bar{a}^{i,j}$, $\bar{c}$ are disjoint.

For $m=1,n$, define f_m as the permutation of A (of order two) interchanging $\bar{a}_1^{i,j}$ with $\bar{a}_m^{i,j}$ for $i,j<\omega$, and taking any other $b\in A$ to itself.

Let $B^*=\{\bar{a}_1^{i\,j}: i,j<\omega\}$.

Now the formula

$$\phi^*(x,y,\bar{z},\bar{R},f_1,\cdots,f_n)=\phi(f_1(x),f_2(x),\cdots,f_n(x),\,f_1(y),f_2(y),\cdots,f_n(y),\bar{c},\bar{R})$$

defines on B^* an equivalence relation with infinitely many infinite equivalence classes. This proves Claim 6A.

CLAIM 6B. *Q_{II} is interpretable by Q_ψ if there are $\phi(\bar{x},\bar{y},r)$, $A,R\in R_\psi(A)$ and $\bar{a}^n\in A(n<\omega)$, such that for every $n<\omega$, $\theta_n=\bigwedge_{m<n}\phi(\bar{x},\bar{a}^m,R)\wedge\neg\phi(\bar{x},\bar{a}^n,R)$ is not pseudo-algebraic.*

PROOF. By the compactness theorem we can assume that each formula θ_n is satisfied by $>2^{\aleph_0}$ pairwise disjoint sequences. Let

$$B=\{\bar{a}_i^m: m<\omega,\,1\leqq i\leqq l(\bar{a}^m)\},\ e=\{\langle\bar{b},\bar{c}\rangle:\bar{b},\bar{c}\in A,\ l(\bar{b})=l(\bar{c})$$

$$=l(\bar{x}),\,(\forall\bar{a}\in B)\ A\models\phi[\bar{b},\bar{a},R]\equiv\phi[\bar{c},\bar{a},R]\}.$$

Then e is an equivalence relation over ${}^{l(\bar{a}^m)}A$. The set of sequences which satisfies θ_n is split into at most $2^{\aleph_0}$ equivalence classes (as $|B|=\aleph_0$), so at least one of them contains $>2^{\aleph_0}$ pairwise disjoint sequences, hence is not pseudo-finite. Clearly for $n\neq m$, a sequence satisfying θ_n and a sequence satisfying θ_m are not equivalent. Thus we get our result by Claim 6A.

CLAIM 6C. *If Q_{II} is not interpretable by Q_ψ then for every $\phi(\bar{x},\bar{y},r)$ there are $m(\phi)<\omega$, and $\chi_{\phi,i}(\bar{x},\bar{z},r)$ $i=1,\cdots,m(\phi)$ such that*

for any $A,R\in R_\psi(A)$ there is $\bar{c}\in A$ which satisfies

1) *$A\models(\forall\bar{x})\bigvee_{i=1}^{m(\phi)}\chi_{\phi,i}(\bar{x},\bar{c},R)$*

2) *$A\models\neg(\exists\bar{x})\,[\chi_{\phi,i}(\bar{x},\bar{c},R)\wedge\chi_{\phi,j}(\bar{x},\bar{c},R)]$ for $i\neq j$*

3) *the sets $S_i=\{\bar{a}: A\models\chi_{\phi,i}[\bar{a},\bar{c},R]\}$ are $\phi(\bar{x},\bar{y},r)$-minimal; moreover for some fixed $m_1(\phi)<\omega$, for no S_i and no $\bar{b}\in A$, do both $\{\bar{a}\in S_i: A\models\phi[\bar{a},\bar{b},R]\}$ and*

$\{\bar{a} \in S_i : A \vDash \neg \phi[\bar{a}, \bar{b}, R]\}$ *contain* $m_1(\phi)$ *pairwise disjoint sequences (we call this property* "$(\phi, m_1(\phi))$*-minimality*").

PROOF. By Claim 6B and the compactness theorem, there is an $m_1(\phi) < \omega$ such that we cannot find $A, R \in R_\psi(A)$, sequences $\bar{a}^n \in A$ for $n < m_1(\phi)$, and a formula $\phi^* \in \{\phi(\bar{x}, \bar{y}, r), \neg \phi(\bar{x}, \bar{y}, r)\}$ such that for each $n < m_1(\phi)$, $\bigwedge_{m<n}[\phi^*(\bar{x}, \bar{a}^m, R) \wedge \neg \phi^*(\bar{x}, \bar{a}^n, R)]$ is satisfied by $\geqq m_1(\phi)$ pairwise disjoint sequences.

Now let η denote a sequence of ones and zeros. Define by induction on l, sequences $\bar{a}_\eta l(\eta) \leqq l$ and formulae $\chi_\eta = \chi_\eta(\bar{x}, \bar{b}_\eta, R)$.

For $l = 0$, η the empty sequence, $\chi_\eta = (\forall x)(x = x)$.

Suppose we have made the definitions for l; let us do so for $l+1$. Let $l(\eta) = l$. If there is an $\bar{a}_\eta \in A$ such that both $\chi_\eta(\bar{x}, \bar{b}_\eta R) \wedge \phi(\bar{x}, \bar{a}_\eta, R), \chi_\eta(\bar{x}, \bar{b}_\eta, R) \wedge \neg \phi(\bar{x}, \bar{a}_\eta, R)$ are satisfied by $\geqq m_1(\phi)$ pairwise disjoint sequences, then choose such $\bar{a}_\eta$; otherwise choose $\bar{a}_\eta$ arbitrarily.

Then if $l(\eta) = l + 1$, define $\chi_\eta(\bar{x}, \bar{b}_\eta, R)$ as follows: $\eta = \langle i(1), \cdots, i(l + 1)\rangle$; then if $i(l + 1) = 0$,

$$\chi_\eta(\bar{x}, \bar{b}_\eta, R) = \chi_{\langle i(1), \ldots, i(l)\rangle}(\bar{x}, \bar{b}_{\langle i(1), \ldots, i(l)\rangle}, R) \wedge \phi(\bar{x}, \bar{a}_{\langle i(1), \ldots, i(l)\rangle}, R)$$

and if $i(l + 1) = 1$, it is the same with $\neg \phi$ instead of ϕ.

By the definition of $m_1(\phi)$, if, e.g., $l(\eta) = 2m_1(\phi) + 2$, then $\chi_\eta(\bar{x}, \bar{b}_\eta, R)$ is $(\phi, m_1(\phi))$-minimal. Clearly the $\chi_\eta(\bar{x}, \bar{b}_\eta, R)$, $l(\eta) = 2m_1(\phi) + 2$ form a partition; and the choice of $\chi_\eta(\bar{x}, z, r)$ does not depend on the particular model. Thus Claim 6C is proved.

CLAIM 6D. *Suppose Q_{II} is not interpretable by Q_ψ. If A is an infinite $R \in R_\psi(A)$, $B \subseteq A$, $\bar{a}, \bar{b} \in A$, and $\bar{a}$ is pseudo-algebraic over $B \cup \{\cdots, \bar{b}_i, \cdots\}$ but not over B, then $\bar{b}$ is pseudo-algebraic over $B \cup \{\cdots, \bar{a}_i, \cdots\}$.*

PROOF. Suppose the conclusion fails. There are $\bar{c} \in B$, and $\phi(\bar{x}, \bar{y}, \bar{z}, r)$ such that $A \vDash \phi[\bar{a}, \bar{b}, \bar{c}, R]$, and $\phi(\bar{x}, \bar{b}, \bar{c}, R)$ is pseudo-algebraic. Say there do not exist m pairwise disjoint sequences in $\phi[A, \bar{b}, \bar{c}, R]$. Let $\theta(\bar{x}, \bar{y}, \bar{z}, R)$ say that $\phi(\bar{x}, \bar{y}, \bar{z}, R)$ and there do not exist m pairwise disjoint sequences in $\phi(A, \bar{y}, \bar{z}, R)$. Since $A \vDash \theta[\bar{a}, \bar{b}, \bar{c}, R]$, $\theta[\bar{a}, \bar{y}, \bar{c}, R]$ is not pseudo-algebraic. For each $n < \omega$, let $\chi_n(\bar{x}, \bar{z}, R)$ say that there are n disjoint sequences $\bar{d}$ such that $\theta(\bar{x}, \bar{d}, \bar{z}, R)$ is satisfied. Thus $A \vDash \chi_n[\bar{a}, \bar{c}, R]$ for all n, and hence $\chi_n(\bar{x}, \bar{c}, R)$ is not pseudo-algebraic.

Now, by the compactness theorem, we can assume that there are $\bar{a}^i$, $\bar{b}^{i,j} \in A$ for $i, j < \omega$ such that

$$A \models \theta[\bar{a}^i, \bar{b}^{i,j}, \bar{c}, R] \text{ for all } i,j,$$

and $\bar{a}^k$, $\bar{a}^l$ (likewise $\bar{b}^{i,k}$, $\bar{b}^{i,l}$) are disjoint for $k \neq l$. By rejecting some $\bar{b}^{i,j}$, we can assume that $\bar{b}^{i,j}$, $\bar{b}^{k,l}$are disjoint unless $\langle i,j\rangle = \langle k,l\rangle$, and also that

$$A \models \theta[\bar{a}^i, \bar{b}^{j,k}, \bar{c}, R] \equiv \theta[\bar{a}^i, \bar{b}^{j,l}, \bar{c}, R]$$

when $i \leqq j$. Further, by Ramsey's theorem, we arrange that the truth value of $\theta[\bar{a}^i, \bar{b}^{j,k}, \bar{c}, R]$ for $i < j$ is independent of i,j.

Now since there are no m pairwise disjoint sequences in $\theta[A, \bar{b}^{m,0}, \bar{c}, R]$, it follows that for all i,j,k, with $i \leq j$, $A \models \theta[\bar{a}^i_, \bar{b}^{j,k}, \bar{c}, R]$ if and only if $i = j$. Thus we get a contradiction as in Claim 6B.

CLAIM 6E. *If $\bar{a} = \langle \bar{a}_1, \cdots, \bar{a}_n\rangle$ is pseudo-algebraic over $B \subseteq A$ in (A,R), then some a_i is algebraic over B in (A,R).*

PROOF. Since $\bar{a}$ is pseudo-algebraic over B, there is a pseudo-algebraic $\phi(\bar{x}, \bar{b}, R)$ ($\bar{b} \in B$), $A \models \phi[\bar{a}, \bar{b}, R]$. Hence there is a finite set $C = \{c_1, \cdots, c_n\}$ such that for any $\bar{a}^1 \in A$, $A \models \phi[\bar{a}^1, \bar{b}, R]$ implies $\{\bar{a}^1_1, \cdots\}$ and C are not disjoint. Without loss of generality n is minimal. Let

$$\theta^1(z_1, \cdots, z_n, \bar{y}, r) = (\forall \bar{x})[\phi(\bar{x}, \bar{y}, r) \rightarrow \bigvee_{i,j} \bar{x}_i = z_j]$$

$$\theta^2(z, \bar{y}, r) = (\exists z_2, \cdots, z_n)\theta^1(z, z_2, \cdots, z_n, r).$$

Clearly for some i, $A \models \theta^2[\bar{a}_i, \bar{b}, R]$. As in Claim 4C we can show that $\theta^2(z, \bar{b}, R)$ is algebraic.

CLAIM 6F. *Assume Q_{II} is not interpretable by Q_ψ. Let $R \in R_\psi(A)$, and for every formula ϕ, let $\chi_{\phi,i}$ $i = 1, \cdots, m(\phi)$, $\bar{c}^\phi$ be as in Claim 6C. Let $C = \{\bar{c}^\phi_i : \phi, i\} \cup \{$elements algebraic over some $\bar{c}^b\}$.*

If $\bar{a}$, $\bar{b} \in A$, $l(\bar{a}) = l(\bar{b}) = n$ and if the following conditions are met:

1) if $\bar{a}_{i_2}, \cdots, \bar{a}_{i_l}$ are algebraic over $C \cup \{\bar{a}_{i_1}\}$, then $\langle \bar{a}_{i_1}, \cdots, \bar{a}_{i_l}\rangle$, $\langle \bar{b}_{i_1}, \cdots, \bar{b}_{i_l}\rangle$ realize the same type over C in (A,R),

2) as in (1), interchanging $\bar{a}, \bar{b}$,

then $\bar{a}, \bar{b}$ realize the same type over C.

PROOF. We prove by induction on n.

For $n = 1$, (1) for $l = 1$ is the conclusion.

Suppose we have proved the claim for n; we shall prove it for $n + 1$. Let $\phi = \phi(x, \bar{y}, \bar{z}, r)$ be a formula, $\bar{c} \in C$.

If each $\bar{a}_i$ is algebraic over $\bar{a}_1$ we are finished. By renaming the $\bar{a}_i$'s we can assume that $\bar{a}_2,\cdots,\bar{a}_l$ are algebraic over $C \cup \{a_1\}$, but $a_{l+1},\cdots,\bar{a}_{n+1}$ are not; $l \leqq n$. Let

$$\bar{a}^1 = \langle \bar{a}_1,\cdots,\bar{a}_l\rangle,\ \bar{a}^2 = \langle \bar{a}_{l+1},\cdots,\bar{a}_{n+1}\rangle,$$

$$\bar{b}^1 = \langle \bar{b}_1,\cdots,\bar{b}_l\rangle,\ \bar{b}^2 = \langle \bar{b}_{l+1},\cdots,\bar{b}_{n+1}\rangle.$$

By (1) and (2), $\bar{b}_2,\cdots,\bar{b}_l$ are algebraic over $\bar{b}_1$, but $b_{l+1},\cdots,\bar{b}_{n+1}$ are not. By Claim 6E, $\bar{a}^2,\bar{b}^2$ are not pseudo-algebraic over, respectively, $\bar{a}^1 \cup C$, $\bar{b}^1 \cup C$.

We must prove that for any $\bar{c} \in C$, $\phi(\bar{x},\bar{y},\bar{z},r)$, $A \vDash \phi[\bar{a}^1,\bar{a}^2,\bar{c},R] \equiv \phi[\bar{b}^1, \bar{b}^2,\bar{c},R]$. By the induction hypothesis, $\bar{a}^i$, $\bar{b}^i$ realize the same type over C. Now we apply the definition of $\bar{c}^\psi$ for $\psi(\bar{y},\bar{x},\bar{z},R) = \phi(\bar{x},\bar{y},\bar{z},R)$ (see Claim 6C).

By Claim 6C (1) there is an i such that $A \vDash \chi_{\psi,i}[\bar{a}^2,\bar{c}^\psi,R]$.

By Claim 6C (2) one of

$$\chi_{\psi,i}(\bar{y},\bar{c}^\psi,R) \wedge \phi(\bar{a}^1,\bar{y},\bar{c},R)$$

$$\chi_{\psi,i}(\bar{y},\bar{c}^\psi,R) \wedge \neg\, \phi(\bar{a}^1,\bar{y},\bar{c},R)$$

(w.l.o.g. the second), is not satisfied by $\geqq m_1(\psi)$ pairwise disjoint sequences. As $\bar{a}^2$ is not pseudo-algebraic over $\bar{a}^1 \cup C$, clearly

$$A \vDash \phi[\bar{a}^1,\ \bar{a}^2,\bar{c},R].$$

Since $\bar{a}^2$ and $\bar{b}^2$ have the same type over C, $A \vDash \chi_{\psi,i}[\bar{b}^2,\bar{c}^\psi,R]$, and since $\bar{a}^1,\bar{b}^1$ have the same type over C, $\chi_{\psi,i}[\bar{y},\bar{c}^\psi,R] \wedge \neg\, \phi(\bar{b}^1,\bar{y},\bar{c}^\psi,R)$ is not satisfied by $\geqq m_1(\psi)$ pairwise disjoint sequences. Hence the above reasoning gives that

$$A \vDash \phi[\bar{b}^1,\bar{b}^2,\bar{c},R]$$

which completes the proof.

Claim 6G. *Suppose Q_{II} cannot be interpreted by Q_ψ. Then there are $n_0,n_1 < \omega$, $\phi(x,y,\bar{z},r)$, $\chi_i(\bar{x}^i,\bar{z},r)$ $i < n_1$ $l(\bar{x}^i) = n^i$ such that $(\exists^{\leqq n_0}x)\ \phi(x,y,\bar{z},r)$ and $\phi(x,x,\bar{z},r)$ and $(\exists^{\leqq n_1}y)\phi(x,y,\bar{z},r)$ hold and for any $A, R \in R_\psi(A)$ there is a $\bar{c} \in A$, such that if $\bar{a},\bar{b} \in A$ $(l\bar{a}) = l(\bar{b}) = n(\psi)$ and if the following conditions are met*

1) *if* $\vDash \phi[\bar{a}_{i_l},\bar{a}_{i_1},\bar{c},R]$ *for* $l = 2,\cdots,k$ *and* $n^i = k$ *then* $A \vDash \chi_i[\bar{a}_{i_1},\cdots,\bar{a}_{i_k},\bar{c},R] \equiv \chi_i[\bar{b}_{i_1},\cdots,\bar{b}_{i_k},\bar{c},R]$,

2) *as in* (1), *interchanging* $\bar{a}$ *and* $\bar{b}$,

then $A \vDash r[\bar{a}] \equiv r[\bar{b}]$.

PROOF. It follows from Claim 6D and 6F and the compactness theorem. (Note that in Claim 6F, we can choose any $\bar{c}^{\phi}$, as long as it satisfies a first-order condition which expresses (1), (2), and (3) of Claim 6C, when we are interested in the formula $r(\bar{x})$ only. We can have one ϕ because the disjunction of algebraic formulae is algebraic and if a is algebraic over B, then for some $n, \phi, \bar{b} \in B$, $A \models (\exists^{\leq n} x)\phi(x, \bar{b}, R)$; hence a satisfies $\theta^1(x, \bar{b}, R) = (\exists^{\leq n} y)\theta(y, \bar{b}, R) \wedge \theta(x, \bar{b}, R)$, and $(\exists^{\leq n} x)\theta^1(x, \bar{b}, R)$ holds.)

PROOF OF LEMMA 6. Assume Q_{II} cannot be interpreted by Q_{ψ}, and we shall interpret Q_{ψ} by Q_P. We use the results and notation of Claim 6G.

Call a, b n-connected (in (A, R), $R \in R_{\psi}(A)$, $\bar{c}$ as in Claim 6G if there are $a = c^0$, $c^2, \cdots, c^n = b$ such that $A \models \phi[c^i, c^{i+1}, \bar{c}, R] \vee \phi[c^{i+1}, c^i, \bar{c}, R]$ for $1 \leqq i < n$. By the remark above, the number of b's n-connected to a is $\leqq k(n) < \omega$ ($k(n)$ depends only on ϕ, ψ and n).

Now choose inductively $A_n \subseteq A$, $n \geqq 1$ such that A_n is a maximal subset of $A - \cup_{i<n} A_i$ with no two 2-connected elements. For $n \geqq k(2) + 2$, A_n is empty, because if $a \in A_n$, then by the definition of A_i, $(i < n)$ there is a $b_i \in A_i$ such that a, b_i are 2-connected. So $> k(2)$ elements are two-connected to A, a contradiction. Now for any $a \neq b \in A_n$, $\phi(A, a, \bar{c}, R)$, $\phi(A, b, \bar{c}, R)$ are disjoint (because if c is in the intersection, then c, a and c, b are 1-connected, hence a, b are 2-connected).

Now it is clear how to define r by permutations and sets. By dividing the A_i's according to $|\phi(A, a, \bar{c}, R)|$, we get $A = \cup_{i<m} A_i$, $a \neq b \in A_i$ implies $\phi(A, a, \bar{c}, R) \cap \phi(A, b, \bar{c}, R) = \emptyset$, and $|\phi(A, a, \bar{c}, R)| = m(i)$. For each i choose permutations of order two $f_1^i, \cdots, f_{m(i)}^i$ such that

$$\phi(A, a, \bar{c}, R) = \{f_j^i(a) : 1 \leqq j \leqq m(i)\}.$$

In view of Claim 6G, we thus represent $R[\in R_{\psi}(A)]$ by the permutations f_j^i, the sets A_i, and the additional sets

$$A_{i,k,l_1\cdots} = \{a \in A_i : A \models \chi_k[f_{l_1}^i(a), \cdots, R]\}.$$

In fact there are only finitely many such possible representations, so by adding a sequence of elements, we can encode, by equalities, the proper case.

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